Embedding algebras into entropic polyquasigroups

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Praha, August 21st 2007

Subreducts of modules

Definition

Algebra (A, Ω) is a reduct of a module $(A, +, 0, R)$ if for each $\omega \in \Omega$ there are $r_i^\omega \in R$ such that

$$\omega(x_1, \dots, x_n) = r_1^\omega x_1 + \dots + r_n^\omega x_n.$$

A subreduct is a subalgebra of a reduct.

Entropic identities

Fact

Each subreduct of a module over a commutative ring is entropic, i.e. it satisfies all identities

$$\mu(\nu(x_1^1,\dots,x_n^1),\dots,\nu(x_1^m,\dots,x_n^m)) \approx \nu(\mu(x_1^1,\dots,x_1^m),\dots,\mu(x_n^1,\dots,x_n^m))$$

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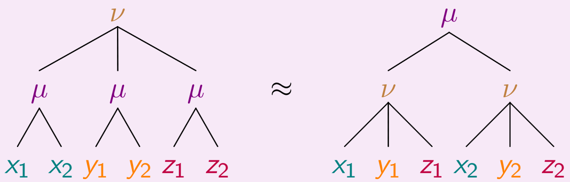
Example of an entropic identity



Cancellative algebras

Cancellation Law

$$\omega(x_1, \ldots, y, \ldots, x_n) \approx \omega(x_1, \ldots, z, \ldots, x_n) \longrightarrow y \approx z$$

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Cancellative algebras:

1. (Quasi)Groups,
2. $(R - \{0\}, \cdot)$, where R is an integral domain,
3. Let M be a R-module and $r_1, \ldots, r_n \in R - \bigcup_{m \in M} Ann(m)$. If

$$\omega(m_1, \ldots, m_n) = r_1 m_1 + \ldots + r_n m_n,$$

then the algebra (M, ω) is cancellative.

Romanowska-Smith theorem

Theorem (Romanowska, Smith)

If entropic idempotent algebra (a mode) is cancellative, then it is a subreduct of a module over a commutative ring.

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Do we really need such strong assumptions? In particular do we need

1. *idempotency?*
2. *all cancellation laws?*

Polyquasigroups

Definition

An algebra (A, Ω) is a polyquasigroup if each translation

$$x \mapsto \omega(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n),$$

where $a_j \in A$ and $\omega \in \Omega$, is bijective.

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Theorem (Sholander, Ježek, Kepka, Stronkowski)

Let $\mathcal{V}$ be a variety of entropic algebras. If an algebra from $\mathcal{V}$ is cancellative, then it is a subalgebra of a polyquasigroup from $\mathcal{V}$.

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1. Embed a cancellative mode (A, Ω) into a mode polyquasigroup (B, Ω).
2. For a basic operation ω of an arity $n > 1$ define

$$\omega_1(x_1, \dots, x_n) = y \quad \text{iff} \quad \omega(y, x_2, \dots, x_n) = x_1 \qquad \text{and}$$
$$\omega_n(x_1, \dots, x_n) = y \quad \text{iff} \quad \omega(x_1, x_2, \dots, y) = x_n$$

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3. Then the operation

$$M(x, y, z) = \omega(\omega_1(x, z, \ldots, z), y, \ldots, y, \omega_n(\omega_1(y, z, \ldots, z), y, \ldots, y, z))$$

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4. (B, Ω, M) is a Mal'cev mode equivalent to an affine space.

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2. for each $\omega \in \Omega$ of arity at least 2

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(A, Ω) is a weak polyquasigroup if the appropriate translations are bijective.

Main result

Theorem

Let A be an entropic and weakly cancellative algebra. There exists an entropic weak polyquasigroup B with A as a subalgebra.

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Let A be an entropic and weakly cancellative algebra. There exists an entropic weak polyquasigroup B with A as a subalgebra. Moreover

1. *A and B satisfy precisely the same identities,*
2. *if A is subdirectly irreducible, then B is subdirectly irreducible as well,*
3. *each homomorphism $A \xrightarrow{h} C$ into an entropic weak polyquasigroup is uniquely factorable as $A \hookrightarrow B \xrightarrow{\tilde{h}} C$.*

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The proof of the first part is based on some technical tricks.
The second part was obtained from the proof in groupoid case, given by J. Ježek and T. Kepka, by a simple translation.

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Easy corollaries

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Proofs.

The same as the proof of Romanowska-Smith theorem. □

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(A,Ω) is weakly^2 cancellative weakly entropic if there is an operation $\omega \in \Omega$ of arity $n > 1$ such that

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2. quasi-identities

$$\begin{aligned}
\omega(y, x_1, \ldots, x_{n-1}) \approx \omega(z, x_1, \ldots, x_{n-1}) &\quad \to \quad y \approx z,\\
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hold in (A,Ω).

A weak definition

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(A, Ω) is weakly[2] cancellative weakly entropic if there is an operation $\omega \in \Omega$ of arity $n > 1$ such that

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hold in (A, Ω).

The definition of of a weakly entropic weak[2] polyquasigroup is analogous.

Not so easy corollaries

Theorem (main theorem improved)

Let A be a weakly^2 cancellative weakly entropic algebra. There exists a weakly entropic weak^2 polyquasigroup B with A as a subalgebra.

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Theorem (Kearnes)

Each weakly^2 cancellative weakly entropic algebra is quasi-affine. In particular it is central.

Difficult corollary

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Each weakly cancellative entropic algebra is a subreduct of a module over a commutative ring.

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Problem

Is each weakly^2 cancellative entropic algebra a subreduct of a module over a commutative ring?

The End

Thank you for your attention :-)